Now f may be determined to fulfill these conditions. For if we consider as known the theorem that the discriminant of an algebraical equation of degree n is an homogeneous function of its n coefficients, of degree $2(n-1)$, we may write the discriminant for $n=3$ in the following form, with undetermined coefficients a:

$$\Delta : a_1+\frac{a_2}{3}p+a_3q+\frac{a_4}{27}p^3+\frac{a_5}{4}p^2+\frac{a_6}{4}p^2q+a_7q^3+a_8q^2+\frac{a_9}{3}q^2p+a_{10}pq.$$

The elimination of p, q, $\partial q/\partial p$ between the four equations

$$S=0,\ \Delta=0,\ \frac{\partial S}{\partial p}=0,\ \frac{\partial \Delta}{\partial p}=0,$$

and equating to zero (identically) the result of substituting $y=x^2$ in the result gives

$$a_1=a_2=a_3=a_5=a_6=a_7=a_9=a_{10}=0,\ a_4=1,\ a_8=\tfrac{1}{4}.$$

Hence the condition for equal roots is

$$\frac{q^2}{4}+\frac{p^3}{27}=0.$$

Reality of Roots in Cardan's Irreducible Case.

Let $$\frac{q^2}{4}+\frac{p^3}{27}=p^2\delta,\ (\delta\neq 0).$$

Then $$q=\pm\frac{2}{\sqrt{27}}[(-p)^{\frac{3}{2}}+\tfrac{3}{2}(-p)^{\frac{1}{2}}\delta]$$

and $$S : x^2+y^2\pm\frac{2}{\sqrt{27}}[(-p)^{\frac{3}{2}}+\tfrac{3}{2}(-p)^{\frac{1}{2}}\delta]x+(p-1)y=0,$$

$$\frac{\partial S}{\partial p} : \pm\frac{2}{\sqrt{27}}[(-\tfrac{3}{2})(-p)^{\frac{1}{2}}-\tfrac{3}{4}(-p)^{-\frac{1}{2}}\delta]x+y=0,$$

and the envelope is

$$x^4+x^2y^2-y^3-x^2y+9x^2y\delta=0.$$

That is

$$y=x^2+\frac{9x^2y}{x^2+y^2}\delta.$$

Hence the ordinate of this envelope is greater than the corresponding ordinate of the primary when δ is positive, and less when δ is negative. Thus the envelope of S is enclosed entirely within the primary, or is entirely enclosed by the primary according as δ is positive or negative. Hence when δ is positive,

$$\frac{q^2}{4}+\frac{p^3}{27}>0,$$

two roots of (1) are imaginary. But when δ is negative,

$$\frac{q^2}{4}+\frac{p^3}{27}<0 \text{ (Cardan's irreducible case)},$$

S intersects P in three real points, so all the roots are real.

§ 2. The Quartic.

The roots of the general quartic

$$x^4+px^2+qx+r=0 \tag{4}$$

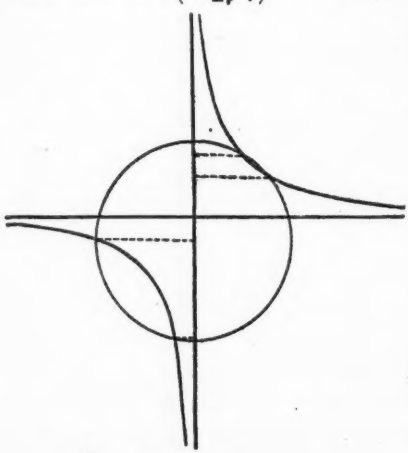
are the abscissas of the intersections of

$$P : xy-1=0,$$
$$S : x^2+ry^2+qy+p=0.$$

The primary (the same for all quartics) is the equilateral hyperbola P. The secondary is a three parameter central conic. Another convenient form is

$$P' : xy-\sqrt{(\epsilon r)}=0,$$
$$S' : x^2+\epsilon y^2+\frac{q}{\sqrt{\epsilon r}}y+p=0,$$

where $\epsilon=1$, -1 as r is positive or negative. If r is positive S' is a circle with center at $\left(0, \frac{-q}{2\sqrt{r}}\right)$. The center of the secondary conic is always on the y axis. The geometrical construction of the roots is now obvious.



II. Scale, ½ in.=unity.

Example: Construct the roots of the quartic

$$x^4-15x^2+10x+24=0.$$

We have

$$P' : xy-2\sqrt{6}=0;$$
$$S' : x^2+y^2+\tfrac{5}{6}\sqrt{6}\,y-15=0.$$

Center $(0, -\frac{5}{12}\sqrt{6})$; radius$=\frac{1}{24}\sqrt{\frac{2305}{6}}$. From the figure [II],

$$r_1=-1,\ r_2=-4,\ r_3=2,\ r_4=3.$$

Harmonic and Equi-Anharmonic Ranges of Roots.

Let us find the envelope of S corresponding to the relation

$$g_2 : \tfrac{1}{12}(p^2+12r)=0,$$

regarding q as invariable. We have from this equation

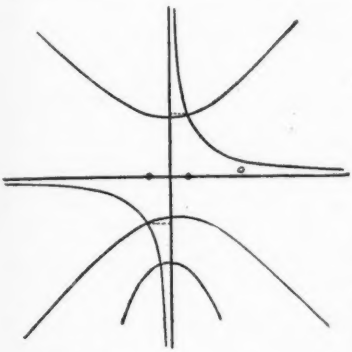
$$S : x^2 - \frac{p^2}{12}y^2+qy+p=0,$$

$$\frac{\partial S}{\partial p} : -\tfrac{1}{6}y^2p+1=0.$$

Hence $p=\dfrac{6}{y^2}$, and the envelope is

$$x^2y^2+qy^3+3=0. \tag{5}$$

We may state the result as follows: *When the secondary conic is tangent to the quartic* (5) *the roots of the biquadratic* (4) *form an equi-anharmonic range.* (See example, Fig. III.)



$x^4+6x^2+x-3=0$

III. Scale, ½ in.=unity.

Next consider p as invariable and let

$$g_3 : \tfrac{1}{6}(pr-\tfrac{3}{8}q^2-\frac{p^3}{36})=0.$$

Then

$$r=\tfrac{3}{8}\frac{q^2}{p}+\frac{p^2}{36},$$

and

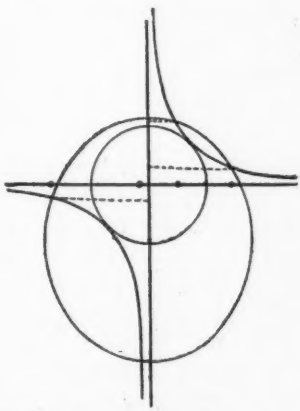
$$S : x^2+(\tfrac{3}{8}\frac{q^2}{p}+\frac{p^2}{36})y^2+qy+p=0,$$

$$\frac{\partial S}{\partial q} : \tfrac{3}{4}\frac{q}{p}y^2+y=0;\ q=-\tfrac{4}{3}\frac{p}{y}.$$

And the envelope is the ellipse

$$\frac{x^2}{\frac{-p}{3}}+\frac{y^2}{\frac{-12}{p}}=1.$$

Thus, *when the secondary conic is tangent to this ellipse the roots of the corresponding biquadratic equation form an harmonic range.* (See example, Fig. IV.)



$4x^4-24x^2+8x+3=0$

IV. Scale, ½ in.=unity.

§ 3. The Quintic.

The roots of the general quintic

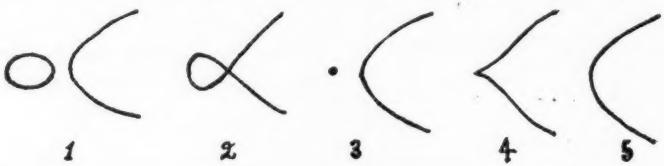
$$x^5+px^3+qx^2+rx+s=0 \tag{6}$$

are the abscissas of the five finite points of intersection of the curves

$$P : xy-1=0,$$
$$S : x^2+sy^3+ry^2+qy+p=0.$$

The second is a Newton divergent parabola.

There are five non-projective classes of the latter curves, as follows:



1. $sx^2=(y-\alpha)(y-\beta)(y-\gamma)$; all real factors, and $\alpha\neq\beta\neq\gamma$.
2. $sx^2=(y-\alpha)(y-\beta)^2$, $\alpha<\beta$; all real factors.
3. $sx^2=(y-\beta)^2(y-\alpha)$, $\beta>\alpha$; all real factors.
4. $sx^2=(y-\beta)^3$; three coincident roots.
5. $sx^2=(y^2+ay+b)(x-c)$; two imaginary roots.

Example: Construct the roots of the quintic equation

$$x^5-21x^3+50x^2-12x-8=0.$$

The secondary curve is

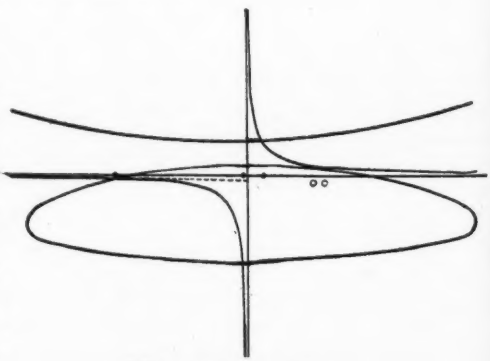
$$x^2-8y^3-12y^2+50y-21=0.$$

That is

$$\frac{x^2}{8}=(y-\tfrac{1}{2})(y-\tfrac{3}{2})(y+\tfrac{7}{2}).$$

Hence the secondary cubic is of type 1. From figure VI we have $r_1=.64+$, r_2 and r_3 imaginary, $r_4=-.3+$, $r_5=-5.4+$.

It is evident that the only classes of the Newtonian parabolas that can intersect the primary hyperbola in five real points are 1 and 2. Hence a necessary condition that all of the roots of a quintic (6) be real is that it be possible to resolve



VI. Scale, $\frac{1}{10}$ in.=unity.

$$sy^3+ry^2+qy+p=0$$

into real factors. That is, that the following relation hold:

$$(7) \qquad (9ps-qr)^2-4(3qs-r^2)(3pr-q^2)\leqq 0.$$

This condition, though necessary, is not sufficient.

§ 4. A Reducible Case of the Sextic.

The sextic

$$(8) \qquad x^6+px^4+qx^3+rx^2+sx+t=0$$

is equivalent to the simultaneous equations

$$xz-1=0;\ x^3+px+tz^3+sz^2+rz+q=0.$$

In these z may be transformed linearly, taking

$$z=t^{-\frac{1}{3}}y+h$$

where h is a root of the cubic

$$(9) \qquad th^3+sh^2+rh+q=0.$$

This gives for the primary and secondary curves

$$P:\ xy+t^{\frac{1}{3}}hx-t^{\frac{1}{3}}=0,$$
$$S:\ x^3+px+y^3+uy^2+vy=0,$$

where $u=t^{-\frac{1}{3}}(3ht+s);\ v=t^{-\frac{1}{3}}(3th^2+2sh+r).$

The secondary cubic passes through the origin and has the asymptote

$$a_1:\ y=-x-\tfrac{1}{3}u.$$

Now S may degenerate into a straight line and a conic. In fact S degenerates if, and only if, one of the two following conditions is fulfilled:

$$d_2:\ v=p;\ u=0,$$
$$d_2:\ u=3\sqrt{-p};\ v=-2p\ (p \text{ necessarily negative}).$$

If $v\neq p$ and $u=0$, the curve S has the origin for a center of symmetry. All these cubics cross the fixed asymptote $y=-x$ at the origin. The origin is a point of inflexion, the inflexional tangent being

$$T:\ y=-\frac{p}{v}x.$$

Thus when $u=0$, *i. e.*, when

$$2s^3-9rst+27qt^2=0,$$

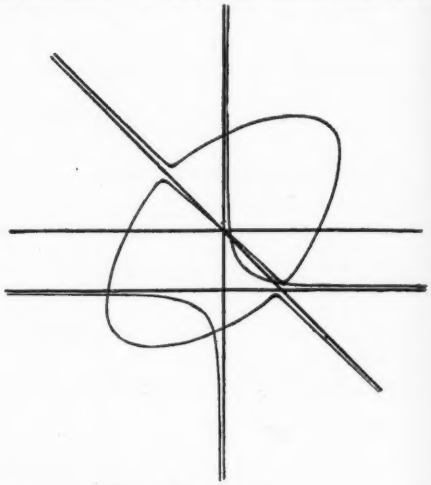
the variation of the secondary, corresponding to changes in the coefficients of the sextic (8), is characterized by the rotation of T around the origin, and when T coincides with the asymptote, S degenerates into straight line and ellipse

$$S: L.C\equiv(x+y)(x^2-xy+y^2+v)=0,$$

i. e., into the asymptote itself and the ellipse $C=0$, origin at center and axis coinciding with the asymptote.

Figure [VII] shows the curves S and P when the sextic (8) is approaching

(10) $x^6-25.9x^4+50.6x^3-2x^2-4x+\frac{1}{3}=0.$



VII. Scale, $\frac{1}{2}$ in.=unity.

When condition d_2 holds we have

$$S: a_1C_1\equiv(x+y+\sqrt{-p})[x^2-xy+y^2+2\sqrt{(-p)}y-\sqrt{(-p)}x]$$

Thus S degenerates into the asymptote, and the ellipse C_1 through the origin, center at the point where the asymptote crosses the y axis, axis coinciding with the asymptote. The sextic equations corresponding to d_1, d_2 may be solved by solving the cubic (9), extracting the roots $\sqrt[3]{t}$, $\sqrt{-p}$, and solving a quadratic and a biquadratic. In fact the sextics corresponding to d_1, d_2 are respectively,

(11) $(x^2-t^{\frac{1}{3}}hx+t^{\frac{1}{3}})[x^4+t^{\frac{1}{3}}hx^3+(t^{\frac{2}{3}}h^2-t^{\frac{1}{3}}+p)x^2-2t^{\frac{2}{3}}hx+t^{\frac{2}{3}}]=0.$

(12) $[x^2+(\sqrt{(-p)}-t^{\frac{1}{3}}h)x+t^{\frac{1}{3}}][x^4+(t^{\frac{1}{3}}h-\sqrt{-p})x^3+(t^{\frac{2}{3}}h^2-2\sqrt{(-p)}t^{\frac{1}{3}}h-t^{\frac{1}{3}})x^2+(2\sqrt{(-p)}t^{\frac{1}{3}}-2t^{\frac{2}{3}}h)x+t^{\frac{2}{3}}]=0.$

FINITE PLANES WITH LESS THAN EIGHT POINTS ON A LINE.

By C. R. MAC INNES, Princeton University.

We get a finite projective plane if we have a finite number of points arranged in sets, called lines, and subject to the following conditions:

1. If A and B are two distinct points there is one and only one line joining them.

2. If α and β are two distinct lines there is one and only one point common to both.

3. There are at least three points on each line.

It follows from these, that we must have the same number of points on all lines. For, having two lines α and β, we can take a point not on either and join it to all the points of α. Each of these joins must also cut β and we therefore have the points on α and β paired.

Also, there must be the same number of points on a line as there are lines through a point. For, having any point A, take a line β not passing through A; join A to each of the points on β. Each point of β has a line through A and each line through A has a point on β.

In short, then, if we have $n+1$ points on a line, we have $n+1$ lines through a point; n^2+n+1 points in the plane, and n^2+n+1 lines in the plane.

Thus, if we have only three points on a line, we have seven points altogether. Denoting these by 0, 1, 2, ..., 6, we get the lines by the following cyclic scheme:

$$0, 1, 2, 3, 4, 5, 6;$$
$$1, 2, 3, 4, 5, 6, 0;$$
$$3, 4, 5, 6, 0, 1, 2;$$

the points in the columns being on a line.

If we leave out one line, we have a plane that might be called a finite Euclidean plane. In this, any two points determine a line, and through any point one and only one line may be drawn not meeting a given line. We have n points on each line, $n+1$ lines through each point, and the lines "parallel" in sets of n. An example of this for $n=3$ is the following: Denote a point by a_{ij} and write the points in the form

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}. \end{matrix}$$

Then the rows, columns, and terms in the ordinary determinant expansion give the twelve lines of the plane.

The existence* of such planes has been proved for the case $n=p^m$, p being prime and m any integer. It has also been proved that there is only one distinct type in each of the cases $n=2$, 3, and 4. Our present problem is for $n=5$ and $n=6$.

Having twenty-five points we are to get all possible Euclidean planes built from them. We may choose any set of parallels and write the points in rows, each row being the five points on one of the lines chosen. By properly choosing the order of the points in the rows, we may fix the columns to give another set of parallels. Finally, by properly choosing the order of the rows we may put into the main diagonal the points of some other line. We now have the scheme:

1.1	1.2	1.3	1.4	1.5
2.1	2.2	2.3	2.4	2.5
3.1	3.2	3.3	3.4	3.5
4.1	4.2	4.3	4.4	4.5
5.1	5.2	5.3	5.4	5.5

in which eleven lines are given by the rows, columns, and main diagonal. We know three of the lines through each point of the main diagonal; to find the others. The only possibilities for lines through 1.1 are

1.1	3.2	2.3	5.4	4.5	
		4.3	5.4	2.5	
		5.3	2.4	4.5	
	.	.	.	.	.
	4.2	2.3	5.4	3.5	
		5.3	2.4	3.5	(A).
			3.4	2.5	
	.	.	.	.	.
	5.2	2.3	3.4	4.5	
		4.3	2.4	3.5	
			3.4	2.5	

Of these we must pick three; they must be one of the following sets:

1.1	3.2	2.3	5.4	4.5	
	4.2	5.3	2.4	3.5	
	5.2	4.3	3.4	2.5	
.	.	.	.	.	.
	3.2	2.3	5.4	4.5	
	4.2	5.3	3.4	2.5	
	5.2	4.3	2.4	3.5	
.	.	.	.	.	(B).

*Veblen and Bussey, *Transactions Mathematical Society*, April, 1906.

3.2	4.3	5.4	2.5
4.2	5.3	2.4	3.5
5.2	2.3	3.4	4.5
.	.	.	.
3.2	5.3	2.4	4.5
4.2	2.3	5.4	3.5
5.2	4.3	3.4	2.5

Doing the corresponding thing for the other points of the main diagonal, we have four other tables matching (B). We have now to pick a set of three from each table so that the fundamental assumptions are satisfied. This can be done in six ways. This gives twenty-six lines; the remaining four are parallel to the diagonal and can be written down immediately as they are determined uniquely by the others. Apparently there are six planes in this case; they can, however, be brought into a one-to-one correspondence, and again we have only one type. The method of establishing the correspondence analytically is not without interest.

We have denoted points by two numbers. If we think of these as coordinates of the points, we find that one of the six planes is such that the points of a line satisfy a first degree equation, using modulus 5 in the algebra. Thus $x=k$ gives the rows, $x-y=0$ the diagonal, and $x-y=k$ the lines parallel to the diagonal. And so on. The lines of this plane are, of course, interchanged by linear substitutions, of which there are 12,000. If to the plane we apply the transformations

$$x=3x_1^3+3\ldots(1),$$
$$x=3x_2^3+3x_2^2+x_2-1\ldots(2),$$
$$x=4x_3^3+2x_3^2+2x_3+3\ldots(3),$$
$$x=x_4^3-x_4^2+2x_4-1\ldots(4),$$
$$x=x_5^3+2x_5^2+3x_5\ldots(5).$$

y being subject to transformations of the same form, we get the five other planes. These transformations, though not linear, are birational, (1) and (2) being inverse to each other, and each of the remaining three being its own inverse. These five, with the identity, form a group.

If we consider the projective plane, it will be unaltered by linear fractional transformations. These form the simple group L.F(3.5) of order 372,000. This order is 31 times as large as in the Euclidean plane, as there are 31 lines, any one of which might be neglected to give the Euclidean plane from the projective one.

If we now try a similar thing for a plane with 36 points, the work proceeds as before till we have written down all the possibilities for lines through (1, 1), (2, 2), and (3, 3). The others need not be considered.

From the lines through (1, 1) we choose a set of four; to these add four through (2, 2) agreeing with them. This can be done in a number of ways. To the eight so found, it is impossible to add four through (3, 3) so that the fundamental assumptions hold. The first assumption breaks down, no matter what combination be tried.

There is therefore no Euclidean plane with 36 points; and consequently no projective plane with only seven points on a line. This includes the result for $n=7$ given in the answer to problem 142, p. 108, Vol. XIV. It is also the result given by Dr. F. H. Safford in answer to problem 132, p. 215, Vol. XIII.

ON CONSTRUCTING A CUBE HAVING A GIVEN RATIO TO A GIVEN CUBE.*

By R. D. CARMICHAEL.

The semi-cubical parabola is capable of a beautiful application to the problem of finding a cube having a given ratio to a given cube. The object of this paper is to give a method of constructing this curve by continuous motion, to apply the locus to the above problem, and also to show how to construct a line numerically equal in length to the cube root of a given line.

The equation of this curve,

$$x^3=py^2, \tag{1}$$

when transformed to polar coordinates with the axis of x as the polar axis, may readily be reduced to

$$\rho\cos\theta=p\tan^2\theta. \tag{2}$$

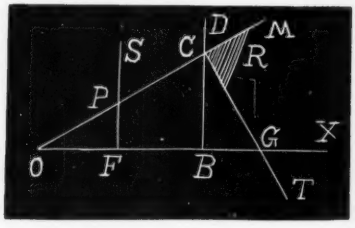


Fig. 1.

To find a construction of the curve by continuous motion we proceed as follows: Let OX (Fig. 1) be the polar axis. Take a distance $OB=p$ and erect BD perpendicular to OX, cutting OM at C, the angle MOX being equal to θ. Draw CG perpendicular to OM at C, intersecting OX in G. From O lay off $OF=BG$, and draw FS perpendicular to OX, intersecting OM in P. P is a point on the semi-cubical parabola.

Proof. Since $OB=p$, $BC=p\tan\theta$. But, since OCG is a right triangle and CB is perpendicular to OG, $OB:BC::BC:BG$. Hence,

*Read before the American Mathematical Society, April, 1907.

(3) $$p.BG=BC^2=p^2\tan^2\theta;\text{ or }BG=p\tan^2\theta.$$

But by the equation of the curve, $p\tan^2\theta=\rho\cos\theta$. But if $OP=\rho$, then $OF=\rho\cos\theta=p\tan^2\theta=BG$. Hence P is a point on the locus.

Now since $OF=BG$ and $OB=p$, $FG=p$. The following construction of the curve by continuous motion is then evident: Let OM and OX be two material lines pivoted at O, OX being fixed to the plane of the paper. Let the material right triangle R, with the material straight line CG attached to it, be fastened to OM so that it may slide freely along OM. Let another material line $FG=p$ be attached to OX so as to slide along it and at the same time let CT pass through a ring at G of FG so as to slide freely. BC is a material line fastened to the plane of the paper at P and perpendicular to OX, OB being equal to p. It passes through a ring at the vertex of the right angle in the triangle R. Finally, FS is a material line fastened to GF perpendicular to it at its extremity F. This intersects OM at some point P. Then let a ring pass around OM and FS at P and let a pencil be placed in this ring. As OM revolves about O the pencil at P will describe the semi-cubical parabola $x^3=py^2$.

We shall now apply this locus to the problem of finding a cube having to a given cube the given ratio of the line m to the line n. For this we shall employ the special case $p=1$. In Fig. 2 lay off $x=OH=$a side of the given cube. Draw $HQ=y$, Q being a point on the locus. Take $OB=-1$ and erect BC perpendicular to OX and cutting QR in C, QR being parallel to OX. Draw OM through C, and then CG perpendicular to OM at C and cutting OX in G. By the same reasoning as in finding equation (3) it follows that

$$BG=\overline{BC}^2,=\overline{QH}^2=y^2.$$

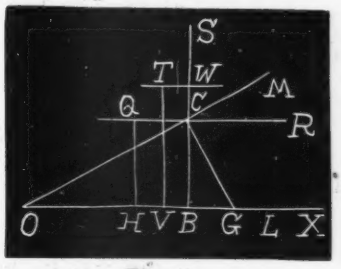


Fig. 2.

But $x^3=y^2$. Hence $BG=OH^3$.

Now find a fourth proportional to m, n, BG, and from B lay off this line, say BL. Extend BC to S and on OL as a diameter describe a semi-circumference cutting BS at W and through W draw a line parallel to OX and cutting the curve in T. From T let fall a perpendicular to OX at V. Then we may repeat the previous reasoning to show that $OV^3=BL$. And therefore, since $OH^3=BG$ and $m:n::BG:BL$, $OH^3:OV^3::m:n$.

We have seen above that $OH^3=BG$, $OH=\sqrt[3]{BG}$. We may thus construct a line numerically equal in length to the cube root of any other line; or, conversely, we may construct a line numerically equal to the cube of a given line.

Remark. It will of course be observed that the problem here solved is a generalization of that of the duplication of the cube, renowned from

antiquity. By this method the ratio of the given cube to the required may be any capable of being expressed as the ratio of two straight lines, and that the problem is not more difficult of solution for one ratio than for another.

Presbyterian College, Anniston, Ala.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

314. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Find the area of the triangle bounded by the lines $l\alpha+m\beta+n\gamma=0$; $l'\alpha+m'\beta+n'\gamma=0$; $l''\alpha+m''\beta+n''\gamma=0$, where α stands for $x\cos\alpha+y\sin\alpha-p$, etc. [See Salmon's *Conic Sections*, 6th Ed.]

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa., and C. N. SCHMALL, 89 Columbia Street, New York City.

Let $A=a(mn'-m'n)+b(nl'-n'l)+c(lm'-l'm)$,

$B=a(mn''-m''n)+b(nl''-n''l)+c(lm''-l''m)$,

$C=a(m'n''-m''n')+b(n'l''-n''l')+c(l'm''-l''m')$.

Then the intersection of (1) and (2) is

$$\frac{\alpha_1}{mn'-m'n}=\frac{\beta_1}{nl'-n'l}=\frac{\gamma_1}{lm'-l'm}=\frac{2\,\Delta}{A}.$$

The intersection of (1) and (3) is

$$\frac{\alpha_2}{mn''-m''n}=\frac{\beta_2}{nl''-n''l}=\frac{\gamma_2}{lm''-l''m}=\frac{2\,\Delta}{B}.$$

The intersection of (2) and (3) is

$$\frac{\alpha_3}{m'n''-m''n'}=\frac{\beta_3}{n'l''-n''l'}=\frac{\gamma_3}{l'm''-l''m'}=\frac{2\,\Delta}{C}.$$

The area of the triangle $(\alpha_1, \beta_1, \gamma_1)$; $(\alpha_2, \beta_2, \gamma_3)$; $(\alpha_3, \beta_3, \gamma_3)$ is

$$\frac{abc}{8\Delta^2}[\alpha_3(\beta_1\gamma_2-\beta_2\gamma_1)+\beta_3(\gamma_1\alpha_2-\gamma_2\alpha_1)+\gamma_3(\alpha_1\beta_2-\alpha_2\beta_1)].$$

Hence the area of the required triangle is

$$\frac{\triangle abc[l(m'n''-m''n')+m(n'l''-l'n'')+n(l'm''-l''m')]^2}{ABC}.$$

Also solved by the Proposer.

315. Proposed by ROBERT E. MORITZ, Ph. D., University of Washington.

Given the area of the segment of a circle of given radius to find the length of the chord.

Solution by G. B. M., ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

I. Let r=radius and $2x$=the length of the chord. Also let A=arc of segment. Then

$$\tfrac{1}{3}x[r-\sqrt{(r^2-x^2)}][+\frac{[r-\sqrt{(r^2-x^2)}]^3}{4x}=A.$$

$$\therefore 169x^5+192r^2x^3-168Arx^2+144(A^2-r^4)x-288Ar^3=0.$$

If A and r are known, x can be found.

II. Let θ=angle of segment at center of circle. Then

$$\tfrac{1}{2}r^2(\theta-\sin\theta)=A,\quad x=r\sin\tfrac{1}{2}\theta.$$

By double position θ is found.

III. $r^2[\sin^{-1}\frac{x}{r}-\frac{x}{r^2}\sqrt{(r^2-x^2)}]=A$. Let $\frac{x}{r}=z$.

$\therefore r^2[\sin^{-1}z-z\sqrt{(1-z^2)}]=A.$

$\therefore \frac{2}{3}z^3+\frac{1}{5}z^5+\frac{3}{28}z^7+\frac{5}{72}z^9+\ldots=A/r^2.$

By reversion of series z is found, then $x=rz$.

316. Proposed by J. STEWART GIBSON, Department of Physics, Wadleigh High School, New York City.

Determine the locus of the vertices of parabolas described by particles thrown off from the circumference of a uniformly revolving wheel.

I. Solution by the PROPOSER.

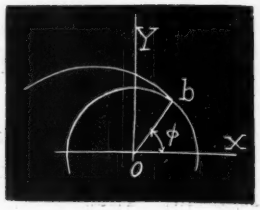
Let r=radius of circle, a=velocity of its periphery, ϕ=angular position of particle b at moment of projection, a_v=vertical component of initial velocity, and a_h=horizontal component of initial velocity. Then $a_v=a\cos\phi$. The height, y_1, to which the particle will rise is (since $h=v^2/2g$),

$$y_1=r\sin\phi+\frac{a^2\cos^2\phi}{2g}. \qquad (1)$$

The time of rise will be $t=\frac{a\cos\phi}{g}$. $a_h=a\sin\phi$.

$\therefore x_1$, the abscissa of the vertex of the parabola, is

$$x_1=-ta_h+r\cos\phi=r\cos\phi-\frac{a^2\sin\phi\cos\phi}{g}. \qquad (2)$$



Transposing and squaring (1),

$$r^2\sin^2\phi=y_1^2-\frac{y_1a^2\cos^2\phi}{g}+\frac{a^4\cos^4\phi}{4g^2};$$

whence $\cos\phi=\pm\sqrt{-\frac{2g(gr^2-y_1a^2}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}}$

and $\sin\phi=\sqrt{(1-\cos^2\phi)}=$

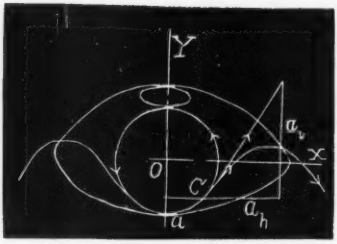
$$\pm\sqrt{1-\left[-\frac{2g(gr^2-y_1a^2)}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}\right]}.$$

Finally, the equation of the required locus is

$$x_1=\pm r\sqrt{-\frac{2g(gr^2-y_1a^2}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}}$$

$$\mp\left[\frac{a^2}{g}\sqrt{1-\left(-\frac{2g(gr^2-y_1a^2)}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}\right)}\right]$$

$$\times\left[\sqrt{-\frac{2g(gr^2-y_1a^2)}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}}\right].$$



The curve has the following peculiarities: It is symmetrical to the vertical axis; is tangent to the circumference at the inferior apex; and also at superior apex by the inclosed loop. Illustrations of the physical formation of the curve are oil drops from a pulley; mud particles from a carriage wheel; water drops from a revolving grindstone; sparks from revolving fireworks, such as "pin wheels," etc.

II. Solution by G. W. GREENWOOD, Dunbar, Pa.

Taking horizontal and vertical axes through the point of projection and in its plane, the path of a particle with initial velocity v, and making initially an angle θ with the vertical, is given by

$$x=v\sin\theta\, t,\quad y=v\cos\theta\, t-\tfrac{1}{2}gt^2.$$

Eliminating t, we get as the equation to the path described,

$$x^2-\frac{2v^2x\sin\theta\cos\theta}{g}+\frac{2v^2\sin^2\theta\, y}{g}=0;\ i.\,e.,\ \left[x-\frac{v^2\sin 2\theta}{2g}\right]^2+\left[y-\frac{v^2\cos 2\theta}{2g}\right]$$

$$=\left[y-\frac{v^2}{2g}\right],$$

which is a parabola whose focus is

$$\frac{v^2\sin 2\theta}{2g},\quad \frac{v^2\cos 2\theta}{2g}.$$

Now taking horizontal and vertical axes through the center of the wheel, and in its plane, the wheel being supposed to revolve clock-wise, the focus of the parabola described by a particle from the point $(-a\cos\theta,\ a\sin\theta)$, a being the radius of the wheel, is given by

$$x=\frac{v^2\sin 2\theta}{2g}-a\cos\theta,\quad y=\frac{v^2\cos 2\theta}{2g}+a\sin\theta,$$

which is the equation of the required locus in terms of the parameter θ.

An excellent solution of this problem was received from G. B. M. Zerr.

CALCULUS.

241. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

Differentiate $y=1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\text{etc.}}}}}$

Solution by J. SCHEFFER, A. M., Hagerstown, Md.; FRANCIS RUST, Allegheny, Pa., and the PROPOSER.

The continued fraction is equivalent to $\frac{1}{2}+\sqrt{(\frac{1}{4}+x)}$.

Hence, $y=\frac{1}{2}+\sqrt{(\frac{1}{4}+x)}$, and $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{(1+4x)}}$.

Also solved by G. B. M. Zerr, G. W. Greenwood, and A. H. Holmes.

242. Proposed by J. H. MEYER, S. J., Augusta, Ga.

A given sphere is to be formed into a solid composed of two equal cones on opposite sides of a common base, in such a manner that its surface may be the least possible. Find the dimensions of the solid, and compare its surface with that of the sphere.

Solution by A. H. HOLMES, Brunswick, Maine.

Of the cones into which the given sphere, radius R, is to be transformed, let x=radius of base, and y=altitude.

Then $\frac{2\pi x^2 y}{3}=\frac{4\pi R^3}{3}$ or $x^2y=2R^3$ and $2\pi x\sqrt{(x^2+y^2)}$=a minimum, or

$x^4+\frac{4R^6}{x^2}$=a minimum.

$\therefore 4x^3=\frac{8R^6}{x^3}$, and therefore $x=2^{\frac{1}{3}}R$ and $y=2^{\frac{2}{3}}R$.

Put S_1=surface of sphere, and S_2=surface of required solid.
Then $S_1:S_2=4:2^{\frac{1}{3}}\sqrt{3}$.

Also solved by G. B. M. Zerr.

MECHANICS.

188. Proposed by H. L. ORCHARD, M. A., B. S.

Spherical bubbles of air are rising in water. Find the relation between radius and velocity.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let R=radius of bubble at surface of water, r=radius of bubble at start at bottom, δ=density of gas in bubble referred to water as unity, w=weight of one cubic inch of water in pounds, h=height of column of water equal to weight of one atmosphere, d=depth of water where bubble starts, v=velocity of bubble at distance s from starting point, bubble starting from rest, x=radius of bubble at distance s from starting point, f=acceleration.

$\therefore \frac{4}{3}\pi R^3 w\delta$=weight of gas in pounds, $\frac{4}{3}\pi R^3 w$=force, in pounds, impelling bubble upwards.

$$\therefore f=\frac{\frac{4}{3}\pi R^3 w(1-\delta)g}{\frac{4}{3}\pi R^3 w(1+\delta)}=\frac{(1-\delta)\,g}{1+\delta}. \quad \therefore v^2=2fs. \quad \text{Also } h+d:h+d-s=x^3:r^3.$$

$$\therefore s=\frac{(x^3-r^3)\,(h+d)}{x^3}. \qquad \therefore v^2=\frac{2f(x^3-r^3)\,(h+d)}{x^3}.$$

$$\therefore \frac{v^2x^3}{x^3-r^3}=2f(h+d)=\frac{2(1-\delta)\ (h+d)g}{1+\delta}.$$

d can be found by either method in Vol. I, page 134.

202. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Three equal, uniform, similar rods AB, BC, CD, freely jointed at B and C, are hung from a point by two equal strings attached at A and D. Find the position of equilibrium.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

By symmetry, the strings, of length l, say, make equal angles with the vertical, as do also the rods AB and DC; denote these angles by θ and ϕ, respectively. The rod BC is horizontal. Denote the length of each rod by a, the weight by w, and the depth of the center of gravity of the system below the point of support by z, the strings being regarded as weightless.

$$z=[w(l\cos\theta+\tfrac{1}{2}a\cos\phi)+w(l\cos\theta+a\cos\phi)+w(l\cos\theta+\tfrac{1}{2}a\cos\phi)]/3w$$
$$=\tfrac{1}{3}[3l\cos\theta+2a\cos\phi].$$

For equilibrium, the value of z must be a maximum.

$$\therefore 0=3l\sin\theta\, d\theta+2a\sin\phi\, d\phi\ldots(1).$$

Also, by horizontal projection,

$$a=2l\sin\theta+2a\sin\phi\ldots(2).$$
$$\therefore 0=l\cos\theta\, d\theta+a\cos\phi\, d\phi\ldots(3).$$

$\therefore 3\tan\theta=2\tan\phi$ (by eliminating $d\theta$ and $d\phi$ from (1) and (2)). This equation, with equation (3), gives the position of equilibrium.

Also solved by G. B. M. Zerr and J. Scheffer.

AVERAGE AND PROBABILITY.

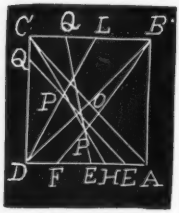
187. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of a given square straight lines are drawn in every possible direction, terminating in the sides of the square. What is the average length of such lines?

II. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $ABCD$ be the given square, side a. P the randon point coordinates (u, v). On account of the symmetry of the square we will confine P to the triangle ADC. Let EQ be the random line through P, $m=\tan\theta=\tan QED$. For the area AOD, E must fall on HF to intersect opposite sides and on AH to intersect adjacent sides. For the area COD, E must fall on

HD to intersect opposite sides and on AH to intersect adjacent sides. For opposite sides $EQ=a\operatorname{cosec}\theta$. For adjacent sides $EQ=(a-u)\sec\theta+v\operatorname{cosec}\theta$. For opposite sides, the limits for AOD are, of v, 0 and $\frac{1}{2}a$; of u, $(a-v)$ and v; of θ, $-\tan^{-1}\left(\frac{a-v}{u}\right)=\theta_2$ and $\tan^{-1}\left(\frac{a-v}{a-u}\right)=\theta_1$; for COD, the limits of u are $\frac{1}{2}a$ and a; of v, u and $(a-u)$; of θ, $-\tan^{-1}\left(\frac{v}{a-u}\right)=\theta_3$ and θ_1. For adjacent sides the the limits of u are 0 and a, of v, 0 and u, of θ, $\tan^{-1}\frac{v}{u}=\theta_4$ and θ_1. Let Δ be the required average length. Then we have for the denominator:



$$D=\int_0^{\frac{1}{2}a}\int_v^{a-v}\int_{\theta_2}^{\theta_1} dv\,du\,d\theta+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\int_{\theta_3}^{\theta_1} du\,dv\,d\theta+\int_0^{a}\int_0^{u}\int_{\theta_4}^{\theta_1} du\,dv\,d\theta$$

$$=\int_0^{\frac{1}{2}a}\int_v^{a-v}\left[\tan^{-1}\left(\frac{a-v}{a-u}\right)+\tan^{-1}\left(\frac{a-v}{u}\right)\right]dv\,du$$

$$+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\left[\tan^{-1}\left(\frac{a-v}{a-u}\right)+\tan^{-1}\left(\frac{v}{a-u}\right)\right]du\,dv$$

$$+\int_0^{a}\int_0^{u}\left[\tan^{-1}\left(\frac{v}{u}\right)-\tan^{-1}\left(\frac{a-v}{a-u}\right)\right]du\,dv$$

$$=\int_0^{\frac{1}{2}a}\left[2(a-v)\log\frac{(a-v)\sqrt{2}}{\sqrt{[(a-v)^2+v^2]}}+\frac{\pi}{2}(a-v)-2v\tan^{-1}\left(\frac{a-v}{v}\right)\right]dv$$

$$+\int_{\frac{1}{2}a}^{a}\left[2(a-u)\log\frac{(a-u)\sqrt{2}}{\sqrt{[(a-u)^2+u^2]}}-\frac{\pi}{2}(a-u)+2u\tan^{-1}\left(\frac{u}{a-u}\right)\right]du$$

$$+\int_0^{a}\left[(a-u)\log\frac{\sqrt{[(a-u)^2+a^2]}}{(a-u)\sqrt{2}}-u\log\sqrt{2}+\frac{\pi a}{4}-a\tan^{-1}\left(\frac{a}{a-u}\right)\right]du$$

$$=\tfrac{1}{4}a^2(\pi+2\log 2).$$

For the numerator we have

$$N=\int_0^{\frac{1}{2}a}\int_v^{a-v}\int_{\theta_2}^{\theta_1} a\operatorname{cosec}\theta\,dv\,du\,d\theta+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\int_{\theta_3}^{\theta_1} a\operatorname{cosec}\theta\,du\,dv\,d\theta$$

$$+\int_0^{a}\int_0^{u}\int_{\theta_4}^{\theta_1}[(a-u)\sec\theta+v\operatorname{cosec}\theta]\,du\,dv\,d\theta$$

$$= a\int_0^{\frac{1}{2}a}\int_v^{a-v} \log\left[\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a-u)}{\sqrt{[u^2+(a-u)^2]}+u}\right]dv\,du$$

$$+ a\int_{\frac{1}{2}a}^{a}\int_{a-v}^{u} \log\left[\left(\frac{v}{a-v}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a-u)}{\sqrt{[(a-u)^2+v^2]}+a-u}\right)\right]du\,dv$$

$$-\int_0^a\int_0^u \left[(a-u)\log\left(\frac{u}{a-u}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}+(a-v)}{\sqrt{[u^2+v^2]}+v}\right)\right]$$

$$+v\log\left[\left(\frac{v}{a-v}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a-u)}{\sqrt{[u^2+v^2]}-u}\right)\right]du\,dv$$

$$= 2a\int_0^{\frac{1}{2}a}\left[v\log\left(\frac{\sqrt{[(a-v)^2+v^2]}+v}{a-v}\right)-(a-v)\log(\sqrt{2}+1)+(a-v)\sqrt{2}\right.$$

$$\left.-\sqrt{[(a-v)^2+v^2]}\right]dv$$

$$+ 2a\int_{\frac{1}{2}a}^{a}\left[2(a-u)\log(\sqrt{2}+1)-(a-u)\log\{\sqrt{[(a-u)^2+u^2]}+u\}\right.$$

$$\left.-u\log\left(\frac{\sqrt{[(a-u)^2+u^2]}+(a-u)}{u}\right)+(a-u)\log)a-u)\right]du$$

$$+\tfrac{1}{2}\int_0^a\left[2au-u^2-a^2\sqrt{2}+(a-u)\sqrt{[(a-u)^2+a^2]}\right.$$

$$\left.-a^2\log\left(\frac{\sqrt{[(a-u)^2+a^2]}-(a-u)}{a(\sqrt{2}-1)}\right)\right]du$$

$$= \frac{a^3}{12}[9(\sqrt{2}+1)\log(\sqrt{2}+1)-4-5\sqrt{2}].$$

$$\therefore \Delta = \frac{N}{D} = \frac{a}{3}\left(\frac{9(\sqrt{2}+1)\log(\sqrt{2}+1)-4-5\sqrt{2}}{\pi+2\log 2}\right).$$

NOTE.—This solution differs from Mr. Heaton's in that Dr. Zerr assumes the lines to be secant lines, that is, lines whose extremities lie on the sides of the square. Mr. Heaton intended the lines to be radial lines, that is, lines the extremities of which are the random point and points on the sides of the square. In both solutions, the same law of distribution has been assumed. ED. F.

188. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Find the average length of a hole at random through a given (a) sphere, (b) cube.

Solution by G. B. M., ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

(a) Let one point remain fixed and be the origin. Then $x^2+y^2+z^2=2ax=d^2$ is the equation to the sphere of radius a.

Let $\sqrt{[2ax-x^2-y^2]}=z'$, $\sqrt{[2ax-x^2]}=y'$, $\triangle$ = average length.

$$\therefore \triangle = \frac{\int_0^{2a}\int_0^{y'}\int_0^{z'} d\,dx\,dy\,dz}{\int_0^{2a}\int_0^{y'}\int_0^{z'} dx\,dy\,dz} = \frac{\sqrt{(2a)}\int_0^{2a}\int_0^{y'}\sqrt{x}\,\sqrt{[2ax-x^2-y^2]}\,dx\,dy}{\int_0^{2a}\int_0^{y'}\sqrt{[2ax-x^2-y^2]}\,dx\,dy}$$

$$= \frac{\sqrt{[2a]}\int_0^{2a}[2ax-x^2]\sqrt{x}\,dx}{\int_0^{2a}\sqrt{[2ax-x^2]}\,dx} = \frac{\frac{64a^4}{35}}{\frac{4a^3}{3}} = \frac{48a}{35}.$$

(b) Let the coordinates of ingress and egress of the hole be for opposite faces $(x, y, 0)$; (u, v, a), and for adjacent faces $(x, y, 0)$; and (a, v, z).

$\sqrt{[(x-u)^2+(y-v)^2+a^2]}=l$, $\sqrt{[(a-x)^2+(y-v)^2+z^2]}=l'$, average length $=p$.

$$\therefore p = \frac{\int_0^a\int_0^a\int_0^x\int_0^y l\,dx\,dy\,du\,dv + \int_0^a\int_0^a\int_0^a\int_0^y l'\,dx\,dy\,dz\,dv}{\int_0^a\int_0^a\int_0^x\int_0^y dx\,dy\,du\,dv + \int_0^a\int_0^a\int_0^a\int_0^y dx\,dy\,dz\,dv}$$

$$= \frac{4}{3a^4}\Big[\int_0^a\int_0^a\int_0^x\int_0^y l\,dx\,dy\,du\,dv + \int_0^a\int_0^a\int_0^a\int_0^y l'\,dx\,dy\,dz\,dv$$

$$= \frac{2}{3a^4}\int_0^a\int_0^a\int_0^z \Big[y\sqrt{[(x-u)^2+y^2+a^2]}$$

$$+[a^2+(x-u)^2]\log\frac{y+\sqrt{[(x-u)^2+y^2+a^2]}}{\sqrt{[a^2+(x-u)^2]}}\Big]dx\,dy\,du$$

$$+\frac{2}{3a^4}\int_0^a\int_0^a\int_0^a \Big[y\sqrt{[(a-x)^2+y^2+z^2]}$$

$$+[z^2+(a-x)^2]\log\frac{y+\sqrt{[(a-x)^2+y^2+z^2]}}{\sqrt{[z^2+(a-x)^2]}}\Big]dx\,dy\,dz$$

$$= \frac{2}{9a^4}\int_0^a\int_0^a \Big[2xy\sqrt{[a^2+x^2+y^2]} + (3a^2y+y^3)\log\frac{x+\sqrt{[a^2+x^2+y^2]}}{\sqrt{[a^2+y^2]}}$$

$$+(3a^2x+x^3)\log\frac{y+\sqrt{[a^2+x^2+y^2]}}{\sqrt{[a^2+x^2]}} - 2a^3\tan^{-1}\left(\frac{xy}{a\sqrt{[a^2+x^2+y^2]}}\right)\Big]dx\,dy$$

$$+\frac{2}{9a^4}\int_0^a\int_0^a 2ay\sqrt{[a^2+y^2+z^2]}+(3z^2y+y^3)\log\frac{a+\sqrt{[a^2+y^2+z^2]}}{\sqrt{[y^2+z^2]}}$$

$$+(3az^2+a^3)\log\frac{y+\sqrt{[a^2+y^2+z^2]}}{\sqrt{[a^2+z^2]}}-2z^3\tan^{-1}\left(\frac{ay}{z\sqrt{[a^2+y^2+z^2]}}\right)\Big]dy\,dz$$

$$=\frac{1}{36a^4}\int_0^a\Big[4a^2y^2-2\pi a^4-2\pi y^4+4y(2a^2+y^2)^{\frac{3}{2}}+2y(a^2+y^2)^{\frac{3}{2}}$$

$$-6y^3\sqrt{[a^2+y^2]}-20\,a^4\tan^{-1}\left(\frac{y}{\sqrt{[2a^2+y^2]}}\right)+4\,y^4\tan^{-1}\left(\frac{y\sqrt{[2a^2+y^2]}}{a^2}\right)$$

$$+4a^4\tan^{-1}\left(\frac{\sqrt{[2a^2+y^2]}}{y}\right)+3a^4\log 2+8(5a^3y+3ay^3)\log\frac{a+\sqrt{[2a^2+y^2]}}{\sqrt{[a^2+y^2]}}$$

$$+2a^2(3a^2+4y^2)\log\frac{y+\sqrt{[a^2+y^2]}}{a\sqrt{2}}+8(3a^4-a^2y^2)\log\frac{y+\sqrt{[2a^2+y^2]}}{a\sqrt{2}}\Big]dy$$

$$=\frac{a}{1620}[286-186\pi+464\sqrt{2}-372\sqrt{3}+768\log(1+\sqrt{2})+1752\log(1+\sqrt{3})$$

$$-936\log 2]=\tfrac{8}{2}\tfrac{5}{7}a,\text{ nearly.}$$

MISCELLANEOUS.

166. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Several equal rectangular boxes are placed in a row with uniform intervals between the boxes and a passageway along one side of the row. Find the least width of the passageway permitting a box to be removed from the row without moving adjacent boxes. This problem arose during the construction of a room for storage batteries.

Solution by the PROPOSER.

Length of box $=PQ=a$; $\angle DAY=\phi=\cos^{-1}(b/c)$. Breadth of box $=PS=b$; $\angle RAY=\theta$. Space between boxes $=AB=c$; $OA=l=c\cot\phi$. Passage in front of boxes $=PW=h$; $OB=m=c\csc\phi$; $AR=n=c\csc\theta-b\cot\theta$.

The complete solution of this problem leads to seven cases corresponding to the possible relations between a, b, c. The desired solution is $h=PW$, and unless otherwise stated; $\phi\neq 0$. The general cases are 1°, $a>c\csc\phi$; 2°, $c\csc\phi>a>c$; 3°, $c>a>c\sin\phi$; 4°, $c\sin\phi>a>0$.

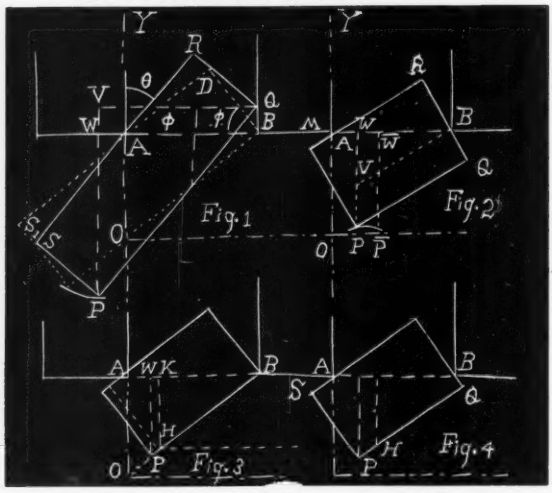
In case 1°, Fig. 1, the box remains partly in the "space" when PW is a maximum. The axes and the auxiliary constant ϕ are determined by the position of the box when Q coincides with B, and the minimum value of y corresponds to the desired value of PW. The path of P is given by

$$x=c-a\sin\theta,\quad y=l+(n-a)\cos\theta-b\sin\theta, \tag{1}$$

in which θ is an auxiliary parameter. The minimum value of y is given by the solution of the transcendental equation

$$c=a\sin^3\theta+b\cos\theta. \tag{2}$$

The instantaneous center of motion of PO is at V, the intersection of the horizontal through Q and the normal to PQ through A. When the box is moving out of the space, V moves toward the left and P toward the right, the minimum y occuring when P crosses the vertical through V. This is the position shown in Fig. 1, from which equation (2) may be verified.



The solution for case 1° is

$$h=l-y=a\cos\theta+b\csc\theta-c\cot\theta, \tag{3}$$

in which θ is the solution of (2).

Fig. 1 shows that h is less than a, so there would be no advantage in withdrawing the box without turning, and then moving it sidewise.

Case 1° is particularly important because the critical point is not at the position BS_1. The limiting case between 1° and 2° occurs when $a=c\csc\phi=OB$, and the solution is then $h=a\cos\phi=ab/c$.

In case 2°, $c\csc\phi>a>c$, the ordinate of P is still decreasing when Q reaches B, as may be shown by the derivative of the second equation in (1) viz.,

$$\frac{dy}{d\theta}=\frac{-1}{\sin^2\theta}(c-a\sin^3\theta-b\cos\theta). \tag{4}$$

Hence the path of P must be studied when SR and QR have sliding contact with A and B, respectively. From Fig. 2,

$$\left.\begin{aligned} x&=b\cos\theta-a\sin\theta+c\sin^2\theta,\\ y&=l-a\cos\theta-b\sin\theta+c\sin\theta\cos\theta, \end{aligned}\right\} \tag{5}$$

$$dy/d\theta = a\sin\theta - c\sin^2\theta + c\cos^2\theta - b\cos\theta = MA - BN. \quad (6)$$

Thus when Q has passed B, the ordinate of P changes and b becomes an increasing one. When a is a little greater than c, P may be in the same vertical as V, the instantaneous center, as seen in Fig. 2, where it is evident that (6) vanishes. This fact does not lead to a solution because P is here at a maximum, so that the provisional solution comes when Q coincides with B, *i. e.*, $\overline{P}\,\overline{W}$, Fig. 2.

On PQ let $HQ = \bar{a}$ be laid off so that

$$c > \bar{a} > c\sin\phi, \quad (7)$$

then from Fig. 3,

$$HK = \bar{a}\,b/c, \quad (8)$$

and it is to be noticed that a further change of position, as shown in Fig. 4, gives by computation, the same value for HK. At the same time (6) shows that during this motion from Fig. 3 to Fig. 4, the ordinate of P increases and then decreases. Since the slope of PQ is less in the latter position, it follows that the ordinate of P is greater in Fig. 4.

If instead of the restriction in (7), $\bar{a}$ be so chosen that

$$c\csc\phi > \bar{a} > c, \quad (9)$$

it follows from (6), since MA does not vanish when R reaches B, that the ordinate of P is still decreasing, hence from (8), with a replacing $\bar{a}$ so as to apply to case 2°,

$$h = PW = ab/c. \quad (10)$$

Thus (10) is the solution for case 2° and gives a value greater than b. In the limiting case between 2° and 3°, *i. e.*, $a = c$, the value is precisely b.

For case 3°, $c > a > c\sin\phi$, Fig. 4 is available if the left boundary is AH instead of SP, so that (8) gives

$$h = HK = ab/c,$$

a result less than b. But a rotation about A must follow, making AH vertical, and now the solution is $h = b$. The limiting case between 3° and 4°, *i. e.*, $a = c\sin\phi$, is that in which the box can just rotate in the space. This case may be considered with case 4°, $c\sin\phi > a$, and the joint solution is $h = b$, as complete rotation is possible.

In the last and trivial case, where $\phi = 0$, b is equal to c, and the solution is $h = a$, because the box cannot be rotated in the space, but must be withdrawn by translation.

PROBLEMS FOR SOLUTION.

ALGEBRA.

289. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\left(\frac{n+1}{(n+1)^2-1}+\frac{1}{3}\cdot\frac{n+3}{(n+3)^1-1}+\frac{1}{5}\cdot\frac{n+5}{(n+5)^2-1}+\ldots\right)$
$+\left(\frac{2}{3}\cdot\frac{1}{n+2}+\frac{4}{15}\cdot\frac{1}{n+4}+\frac{6}{35}\cdot\frac{1}{n+6}+\ldots\right)=\frac{n-1}{(n-1)^2-1}+\frac{1}{3}\cdot\frac{n-3}{(n-3)^2-1}$
$+\frac{1}{5}\cdot\frac{n-5}{(n-5)^2-1}+\ldots+\frac{1}{l}\cdot\frac{n-l}{(n-l)^2-1}$, n being any odd integer greater than 1 and $l=n-2$.

290. Proposed by G. I. HOPKINS, M. A., Manchester, New Hampshire.

A and B are 45 miles apart and travel towards each other. A goes one mile the first day, three miles the second day, five miles the third day, and so on. B goes two miles the first day, four the second, six the third, and so on. In how many days will they meet? What interpretation is to be placed upon the negative value of n?

GEOMETRY.

221. Proposed by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Texas.

ABC is an isosceles triangle. Through any point P in its plane draw a line $PSRT$ cutting the sides AC, CB, AB in the points S, R, and T, respectively (R between B and C), so that the segments CS and BT shall be equal.

222. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

Find all surfaces such that the normal lengths intercepted by the three coordinate planes are in constant ratios for all points.

223. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

S, S' are the foci of two co-vertical parabolas A and B, the axes of which are at right angles. Draw the circle K on SS' as diameter. K is cut in D and E by a straight line parallel to the axis of A such that S' lies midway between it and that axis. Show that the lines $S'D$, $S'E$ are parallel to the two tangents to A which are normals to B.

CALCULUS.

245. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Prove or disprove:

$$\int_0^\infty \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}}=2F'(k)+\sqrt{(-1)}.F'[\sqrt{(1-k^2)}],$$

Legendre's notation, $0<k<1$.

246. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

Derive Taylor's Series by the use of the formula for successive integration by parts, and nothing else.

MECHANICS.

207. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A portion of a parabola is bounded by the curve, the axis and an ordinate. A circle is inscribed to the figure which is regarded as a plane lamina. The area of the inscribed circle is now punched out. Find the centroid of what is left.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

148. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find all the multiply perfect numbers of n different prime factors and of multiplicity $n-1$.

AVERAGE AND PROBABILITY.

191. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random lines cut a given circle. What is the chance that they intersect within the circle?

192. Proposed by A. H. HOLMES, Brunswick, Maine.

In the game of baccarat the dealer and each side of the table have two or three cards. The object is to get as near nine as possible, and tens and court cards do not count. If the two first cards dealt do not together amount to five, the player asks for another. If above five he does not. When the two cards amount to exactly five would the chances of the hand be bettered or diminished by drawing a third card, and how much?

MISCELLANEOUS.

173. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

If n is odd, prove the following: $\pm 1=[(-1)^{1/n}+(-1)^{-(1/n)}][(-1)^{2/n}+(-1)^{-(2/n)}][(-1)^{3/n}+(-1)^{-(3/n)}]\ldots[(-1)^{(n-1)/2n}+(-1)^{-(n-1)/2n}]$

$\pm\sqrt{n}(-1)^{(n-1)/4}=[(-1)^{1/n}-(-1)^{-(1/n)}][-(-1)^{2/n}-(-1)^{-(2/n)}][(-1)^{3/n}-(-1)^{-(3/n)}]\ldots[(-1)^{(n-1)/2n}-(-1)^{-(n-1)/2n}]$.

174. Proposed by L. E. DICKSON, Ph. D., Associate Professor of Mathematics, The University of Chicago.

By a linear transformation with integral coefficients modulo 2, reduce $\Sigma x_i^2+\Sigma x_i x_j$ $(i, j=1, \ldots, 2m; i<j)$ to a canonical form in which the variables are separated into pairs.

NOTES AND NEWS.

A regular meeting of the Southwestern Section of the American Mathematical Society will be held in the buildings of Washington University, St. Louis, on Saturday, November 30, 1907. S.

The next annual meeting of the American Association for the Advancement of Science will be held in Chicago during the Christmas Holidays. Two joint sessions of Sections A and O, Mathematics and Engineering, with the Chicago Section of the American Mathematical Society will be held on Monday afternoon and Tuesday morning, December 30 and 31. The subject for discussion is: "The Teaching of Mathematics to Students of Engineering." Prominent engineers and mathematicians will take part in the program. S.

The sixth annual meeting of the Central Association of Science and Mathematics Teachers will be held in the building of the McKinley High School at St. Louis, Mo., November 29 and 30, 1907. The Mathematics Section will hold sessions Friday afternoon and Saturday morning, at which the reports of two important committees will be presented, one on the teaching of algebra, and the other on the teaching of geometry. The leading speakers in the discussion of these reports will be Professor Florian Cajori, Colorado College; Professor E. R. Hedrick, University of Missouri; Professor G. B. Halsted, Greeley, Colo.; Professor G. C. Shutts, Whitewater, Wis.; W. W. Hart, Shortridge High School, Indianapolis; and C. W. Newhall, Shattuck School, Faribault, Minn. S.

Daniel A. Murray, Ph. D., has recently accepted the chair of Mathematics in McGill University. The title of the latest of his many popular college text books has been changed from "Practical Mathematics" to "Essentials of Trigonometry." OLIPHANT.

Readers of the MONTHLY may secure a year's subscription to the *Technical World Magazine* (regular price, $1.50) and Finkel's *Mathematical Solution Book* (regular price, $2.00), for $2.25 for the magazine and book. For $4.00, one year's subscription to the MONTHLY will be included. *The Cosmopolitan* and the MONTHLY for one year for $2.70.

BOOKS.

A First Course in the Differential and Integral Calculus. By William F. Osgood, Ph. D., Professor of Mathematics in Harvard University. 12mo. Cloth, xv+423 pages. Price, $2.00, net. New York: The Macmillan Co.

Important features, among others, in the treatment of the Calculus as carried out in

this work are, the simplicity, the clearness, and the directness with which the principles underlying the Calculus are set forth and the splendid applications and illustrations of these principles in the solution of problems in physics and mechanics. Since the ideas underlying the Calculus are nowhere brought out more clearly than in the application of its principles to the study of curves and surfaces, in Mechanics, and in definite integrals with their applications to Geometry, Physics, and Astronomy, these subjects are taken up at an earlier stage than is usually customary. Thus, for example, *curvature* is taken up in chapter VII, p. 134; *definite integrals*, Chapter IX, p. 153, and *mechanics*, Chapter X, p. 190. Chapter XVIII deals with double integrals, and Chapter XIX with triple integrals. Here the author uses a notation which should be followed by all writers on the subject, viz., for example, for

$$\int\int\int f(x, y, z)\,dx\,dy\,dz,$$

the notation

$$\int dx \int dy \int f(x, y, z)\,dz$$

is used. By this notation there is no ambiguity as to the order of integration. The book contains many valuable features too numerous to mention in the brief space at our disposal. It is very attractively printed and bound, and the selection of problems is most commendable. F.

High School Algebra. Elementary Course. By H. E. Slaught, Ph. D., Assistant Professor of Mathematics in the University of Chicago, and N. J. Lennes, M. S., Instructor in Mathematics in the Wendell Philips High School. 8vo. Cloth, xii+297 pages. Chicago: Allyn and Bacon.

As stated in the preface, the important features of this text-book are: (1) Algebra is here vitally and persistently connected with arithmetic; (2) the enunciation of the principles of algebra in eighteen short sentences; (3) the solution of problems rather than the construction of purely theoretical doctrine as an end in itself; and (4) the determination of the order of the topics and the inclusion of the order of the topics, and the inclusion and exclusion of subject matter by the main purpose of the course itself.

The book is based on true scientific and pedagogical principles. Great care is taken in laying the foundation of the subject. The problems are drawn, for the most part, from the experiences of every-day life and are of a nature easily within the comprehension of the beginner. From a pedagogical point of view the book is all that can be desired. It seems that all pedagogical requirements have been satisfied not only in this work but perhaps in several of its predecessors, and that the attention of teachers of mathematics in the higher institutions of learning be now directed to, what seems to me, to be the true cause of the lamentable state of elementary mathematical teaching in this country. The cause of poor teaching is not so much the lack of teachable text-books in the hands of the pupils as the lack of teachable teachers into whose hands the pupils are committed. Is it not a fact that the teaching of Algebra and Geometry in the great majority of high schools and academies is intrusted to the merest arithmetical tyros, teachers whose thoughts in regard to mathematics are as dark and confused as are those of a savage respecting the laws of the universe. The most noteworthy progress in the teaching of elementary mathematics will be obtained when teachers who have no more interest in mathematics than to make the perfunctory teaching of it a means to gain a livelihood are crowded to the rear and their places taken by the real, earnest, enthusiastic, and enlightened teacher of mathematics. F.

Plane and Solid Geometry. By Isaac Newton Failor, Principal of the Richmond Hill High School, New York City. 12mo., 420 pages. $1.25, net. New York: The Century Co.

The author has aimed to present to the educational public a work on Geometry that should be both teachable and practicable. In the earlier parts of the book most corollaries

are proved and references and postulates are quoted in full. This is necessary in order to give the beginner a notion of what is required to be done. The demonstrations are so arranged that no page needs be turned to read them. A change which, to my mind, does not add to the attractiveness of the book, is that the theorems are set in ordinary long primer type instead of in italics or black-faced type. The book contains a very large collection of exercises well suited to call out the powers of the student.

The publishers have done all that is possible to make the mechanical features of the book first class. F.

Text-book of Mechanics. By Louis A. Martin, M. E. (Stevens), A. M. (Columbia), Assistant Professor of Mathematics and Mechanics in Stevens Institute of Technology. Vol. II, Kinematics and Kinetics. 12mo. Cloth. xiv+214 pages. 91 figures. Price, $1.50, net. New York: John Wiley and Sons.

This volume completes the author's elementary course in Mechanics, the intention of which course is to prepare the student for courses in Applied Mechanics, and to lay a solid foundation for the study of more difficult works. The study of this volnme requires a knowledge of Analytical Geometry and the Calculus. There are many exercises, the solution of which will enable the student to gauge his own knowledge of the subject as he pursues his course. The book is neatly printed and bound. F.

The Elements of Plane and Spherical Trigonometry. By Edwin S. Crawley, Ph. D., Thomas Scott Professor of Mathematics in the University of Pennsylvania. New and Revised Edition. Entirely rewritten. 8vo. Cloth, v+186 pages. Price, $1.25. Philadelphia: Published by the Author.

In this new edition, important changes and additions have been made. Of these, we note the addition of trigonometric equations and elimination, trigonometric series, and hyperbolic functions. Also some additions have been made in the discussions of lines and circles. Thus, some properties of the nine-points circle have been introduced and the determination of the Brocard points. The book concludes with a prief application of trigonometry to Astronomy. The typography of the book is first class and the binding and paper are excellent. F.

Computation and Mensuration. By P. A. Lambert, M. A., Professor of Mathematics in Lehigh University. 8vo. Cloth, ix+92 pages. Price, $0.80. New York: The Macmillan Co.

This work is divided into ten chapters, the first of which deals with Approximate Computation; the second, with Graphic Computation; the third, with the Method of Coordinates; the fourth, with Volumes of Solids Bounded by Planes; the fifth, Computation and Use of Trigonometric Functions; the sixth, with Computation and Use of Logarithms; the seventh, with Limits; the eighth, with Graphic Algebra; the ninth, with Areas Bounded by Curves; and the tenth, with Volumes of Solids.

The aim of the work is to give the student a training in the application of the knowledge gained in the secondary school mathematics, and is intended to come at the close of the secondary school course or at the beginning of the college course.

The work is well conceived and will, if properly used, serve to increase the student's power and enable him to carry on his work in the college with interest and pleasure. F.

11

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LIBRARY AIDS TO MATHEMATICAL STUDY.

By DR. G. A. MILLER, University of Illinois.

Herr Valentin of Berlin who has been working on a general mathematical bibliography for more than twenty years estimates that the total number of different mathematical works is about 35,000 and that about 95,000 mathematical articles have appeared in the various periodicals.* Morever, the amount of this literature is growing at an increasing rate of speed so that it appears likely that during the next forty years there will be a larger addition to the mathematical literature than the total amount which has appeared up to the present time. In fact, this is a very conservative estimate, since such a work as the *Jahrbuch der Fortschritte der Mathematik* chronicles annually about 2000 books and articles in pure mathematics in addition to a large number in closely allied subjects.

One of the first questions which confronts the student is the relative importance of periodic and non-periodic publications. In general it must be said that these supplement each other and that the advanced student needs both. As the books generally co-ordinate the results obtained by many different writers, broad views can usually be more easily obtained from books than from the separate articles, but these broad views are tinged by the peculiar bent of the author's mind and they naturally do not exhibit the clearness in detail which the student would have obtained by studying the authorities himself. This is especially true of the newer subjects where the number of books is comparatively small and where progress is generally so rapid that the books are deplorably behind the times.

The history of mathematics furnishes a good illustration of the point in question. When the first edition of the first three volumes of Cantor's *Vorlesungen über Geschichte der Mathematik* appeared, it was commonly regarded as authoritative even in nearly all of the details. It was to a large extent instrumental in arousing a more general interest in the history of mathematics and marked the beginning of an unusually active period in historical investigations, so that it is becoming much more difficult for one man

*Felix Mueller, *Bibliotheca Mathematica*, Vol. 7 (1907), p. 416.